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Excitations in hot non-commutative theories

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Abstract: We study the dispersion relation for scalar excitations in supersymmetric, non-commutative theories at finite temperature. In $\mathcal{N} = 4$ Yang-Mills the low momentum modes have superluminous group velocity. In the massless Wess-Zumino model the minimum of the dispersion relation is at non zero momentum for temperatures above $T_0 \approx (g\theta)^{-1/2}$. We briefly comment on $\mathcal{N} = 2$ Yang-Mills at finite density.

Keywords: Superstrings and Heterotic Strings, Space-Time Symmetries.

JHEP09(2000)027

Contents

1. Introduction

Field theories on non-commutative space have interesting properties. In particular, infrared divergences appear whose origin are ultraviolet degrees of freedom circulating in loops. This phenomenon implies a surprising mixing between the ultraviolet and infrared degrees of freedom of the theory [1]. The study of infrared singularities appearing in perturbation theory has been an active field of research recently [2]–[25].

In this paper we will study non-commutative field theories at finite temperature and density. In particular we will study the dispersion relation of scalar fields. The behaviour of hot non-commutative field theories has been investigated in [26]–[29]. At high temperature, when the characteristic thermal wavelength of the particles becomes smaller than the radius of the Moyal cell, $T \gtrsim \theta^{-1/2}$, one observes a suppression of the contribution from non-planar graphs with respect to the planar ones. This suggests that there is a reduction of the number of degrees of freedom running in non-planar graphs [27, 28].

The study of the dispersion relation is relevant because it directly addresses the question of how the physical degrees of freedom of non-commutative field theories differ from their commutative counterparts. It is by now well-known that at zero temperature and density the one loop dispersion relation gets modified by terms showing a pole like infrared singularity [1] which is typically of the form

$$\omega^2 = p^2 + m^2 + \frac{c}{|\tilde{p}^2|} \,. \tag{1.1}$$

Here we used the notation $\tilde{p}^\mu = \theta^{\mu\nu} p_\nu$ and c is a model dependent constant, $c = \mathcal{O}(g^2)$. These infrared poles have their origin in the regularization of the quadratic divergences in non-planar diagrams by the Moyal phases. If the external momentum flowing into the diagram vanishes, the original divergence is recovered but disguised as an infrared divergence. In supersymmetric theories the quadratic divergences are cancelled by Bose-Fermi degeneracy ($c = 0$). In a heat bath supersymmetry is broken. The one loop dispersion relations are then bound to be non-trivial while the infrared behaviour is under control in perturbation theory. It is the aim of this paper to investigate these modifications and their consequences in non-commutative field theories.

In section 2, we discuss the scalar one loop dispersion relation in $\mathcal{N} = 4$ non-commutative Yang-Mills at finite temperature. We find that for all temperatures the group velocity exceeds the speed of light for soft momenta in the non-commutative direction. We study the problem of wave propagation in the limit of small momenta. These modes obey the dispersion relation of the linearised Korteweg-deVries equation. The wavefront of a disturbance travels faster than light. It is perhaps surprising that the speed of light is no more a barrier. This is per se not inconsistent since Lorentz symmetry is absent in non-commutative space.

In section 3, we analyse the non-commutative version of the Wess Zumino model. For sufficiently high temperature a dip appears in the dispersion relation. More precisely, the lowest energy mode has non-vanishing momentum in the non-commutative directions. At high temperatures and small momenta, the group velocity again exceeds the speed of light. We furthermore argue that the dip makes Bose-Einstein condensation impossible.

In section 4 we analyse the effects of non-commutativity on field theories at finite density. We will investigate $\mathcal{N} = 2$ supersymmetric gauge theory with a chemical potential μ for the gauginos. The dispersion relation for the scalars at one loop is similar to the one encountered in section 2. The role of temperature is now played by the chemical potential.

2. $\mathcal{N} = 4$ Yang-Mills

We will work in four-dimensions and consider only space non-commutativity. Without loss of generality we take

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}\,, \tag{2.1}$$

with $\theta^{12} = -\theta^{21} = \theta$ and $\theta^{\mu\nu} = 0$ otherwise. The algebra of functions on this space is defined through the star product

$$(f * g)(x) := \lim_{y\to x} e^{\frac{i}{2}\theta^{\mu\nu}\partial^x_\mu\partial^y_\nu} f(x)g(y)\,, \tag{2.2}$$

where here x^μ are taken to be ordinary c-numbers.

The non-commutative version of $\mathcal{N} = 4$ Yang-Mills [28] is obtained by substituting the star product in commutators, or equivalently, replacing commutators by the so-called Moyal bracket

$$\{f, g\} = (f * g)(x) - (g * f)(x)\,. \tag{2.3}$$

In a non-commutative gauge theory with gauge group $\mathrm{U}(N)$ the Moyal phases derived from the star product cancel in the $\mathrm{SU}(N)$ part at the one loop level [1, 21]. In others words there are no non-planar contributions from particles in the adjoint of $\mathrm{SU}(N)$ running in the loop and the only non-planar one loop graph stems from the $\mathrm{U}(1)$ factor. For this reason we limit ourselves to the study of a $\mathrm{U}(1)$ $\mathcal{N}=4$ gauge theory. The spectrum of the theory consists of six scalars, four Majorana fermions and a vector field. The corresponding Feynman rules can be found in [2, 5].

We will study the dispersion relation of the $\mathcal{N}=4$ scalars at finite temperature and one loop level. We will comment later on the modifications for fermions and gauge bosons. The scalar self-energy is given by

$$\Sigma_T = 32g^2 \int \frac{d^3k}{(2\pi)^3} \frac{\sin^2 \frac{\tilde{p}\cdot k}{2}}{k} \left(n_B(k) + n_F(k)\right) + g^2 P^2 \bar{\Sigma}\,. \tag{2.4}$$

We use uppercases for the four momenta and lowercases for the spatial components, $P^2 = p_0^2 - p^2$. Also $\tilde{p}k \equiv \theta(p_1 k_2 - p_2 k_1)$ and we refer to momenta along the non-commutative directions as transverse momenta, with $|\tilde{p}^2| = \theta^2(p_1^2 + p_2^2) = \theta^2 p_\perp^2$.

The first term in (2.4) vanishes at $T=0$ because of supersymmetry. The second term contributes to the finite temperature wave-function renormalization of the scalar field. It affects the position of the pole only to $\mathcal{O}(g^4)$ and we will drop it in the sequel. A subtlety should be mentioned. The zero temperature planar contribution to $\bar{\Sigma}$ is typically logarithmically divergent. As a consequence the non-planar contribution can give raise to an infrared logarithmic singularity. For a massless theory we will have

$$\bar{\Sigma}^{np}_{T=0} \sim g^2 P^2 \log \frac{1}{\tilde{p}^2 P^2}\,. \tag{2.5}$$

If such a term is present[1], the perturbative expansion breaks down for non-perturbatively small transverse momenta, $\mathcal{O}(e^{-1/g^2})$ [1]. The phenomena we will find in this and the next sections do not depend on entering into this region. Thus in the following we will ignore contributions to the self-energy proportional to g^2P^2.

Using the relation $\sin^2 \frac{\tilde{p}k}{2} = \frac{1}{2}(1 - \cos \tilde{p}k)$ we can separate the planar and non-planar contributions to the self-energy. The dispersion relation simply becomes

$$\omega^2 = p^2 + 2g^2T^2 - \frac{4g^2T}{\pi\tilde{p}} \tanh \frac{\pi\tilde{p}T}{2}\,, \tag{2.6}$$

[1] This term is gauge dependent. In ordinary $\mathcal{N}=4$ gauge theories both quadratic and logarithmic divergences are absent if a convenient gauge is chosen [30]. One can speculate that in the non-commutative version of $\mathcal{N}=4$ there must also exist a gauge in which the theory is smooth in the infrared [5].

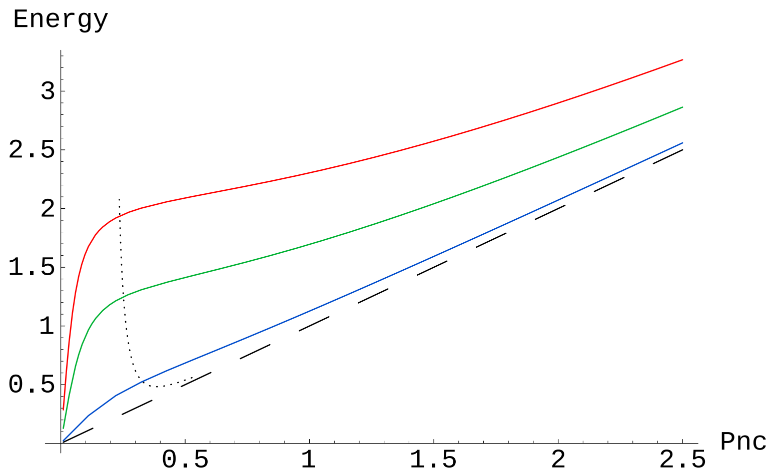

Figure 1: Dispersion relation for scalars in $\mathcal{N}=4$ Yang-Mills for different temperatures. The momentum p is taken to lie entirely in the non-commutative directions. The dashed line shows the light cone $\omega = p$. The dotted line shows the momentum p_c below which the group velocity $\partial\omega/\partial p$ is bigger than one.

where $\tilde{p} \equiv |\tilde{p}|$. The qualitative features of the dispersion relation, which is plotted in figure 1, are easy to understand. The hyperbolic tangent arises solely from the non-planar contribution to the dispersion relation. For large transverse external momenta the non-planar contribution is subleading with respect to the planar one,

$$\omega^2 \approx p^2 + 2g^2T^2 - 4g^2\frac{T}{\pi\tilde{p}}\,, \qquad T\tilde{p} \gg 1\,. \tag{2.7}$$

The second term comes from the planar diagrams and gives a mass to the scalar excitations. The subdominant term linear in T is quite interesting. It arises solely from soft bosons in non-planar diagrams. These are modes with characteristic momentum $k \ll T$ and large occupation number, $n_B \approx T/k \gg 1$,

$$\Sigma_{np} \sim \int d^3k\frac{1}{k}\cos\tilde{p}k\frac{T}{k} \sim \frac{T}{\tilde{p}}\,. \tag{2.8}$$

The ultraviolet catastrophe of usual space-time does not arise as long as the non-commutativity scale, or more precisely $\tilde{p}$, is kept non-zero. This is yet another manifestation of the UV/IR mixing of non-commutative field theories: to leading order at high temperature, the non-planar contribution is effectively purely classical [28]. At low traverse external momenta, the non-planar contribution tend to cancel the planar one,

$$\omega^2 \approx p^2 + 2g^2T^2 - 2g^2T^2 + \frac{g^2\pi^2T^4}{6}\tilde{p}^2\,, \qquad T\tilde{p} \ll 1\,. \tag{2.9}$$

For zero external transverse momentum the interaction switches off. The theory becomes a free, gapless U(1) gauge theory with $\omega^2 \approx p_3^2$.

Let us consider now the case where the momentum lies along the non-commutative directions, $p^2 = p_\perp^2$. Since $\omega(0) = 0$ and for large p, $\omega(p) \approx \sqrt{p^2 + 2g^2T^2}$, which lies above the lightcone, there is a region in between with $\partial\omega(p)/\partial p > 1$. Thus the group velocity must exceed the speed of light for small transverse momenta![2] From (2.9) we find

$$\omega^2 \approx \left(1 + \frac{g^2\pi^2T^4\theta^2}{6}\right) p^2 \,. \tag{2.10}$$

The low momentum excitations are massless, but propagate with an index of refraction $n = p/\omega$ smaller than one. Because the interactions switch off at low momenta, we expect these modes to be long-lived.[3] The dispersion relation for several temperatures is plotted in figure 1. The dashed line shows the light cone $\omega = p$. Below some momentum p_c the group velocity exceeds the speed of light. For high temperature,

$$p_c \approx \sqrt{\frac{\sqrt{2}}{\pi\theta}} g \,. \tag{2.11}$$

Let us emphasise that these qualitative features should be quite general and not an artifact of our one loop approximation, as they simply arise from the fact that the theory is non-interacting at zero transverse momentum and develops a mass gap otherwise. Fermions and gauge bosons dispersion relations are involved at finite temperature. However, it should be clear that these particles share the same qualitative properties as the scalar modes. For instance, the Debye mass of longitudinal gauge bosons is vanishing in the limit of zero transverse momentum.[4]

We now investigate the consequences of the dispersion relation (2.6) for wave propagation. Imagine that some disturbance of the scalar field is created in the thermal bath at time $t = 0$. An elementary solution can be written in the form

$$\Phi(x,t) = \int_{-\infty}^{\infty} \left(F(k)e^{i(kx-\omega t)} + F^*(k)e^{-i(kx-\omega t)}\right) dk \,, \tag{2.12}$$

where $F(k)$ is determined by an initial condition at $t = 0$. To simplify matters we will consider only a one dimensional problem with momentum pointing in a

[2]By small transverse momenta we mean small but still outside the exponentially small region where logarithmic terms as (2.5) could become important.

[3]At one loop the self-energy correction we are considering is real since it is essentially a tadpole contribution. It depends of the external momentum only because of the star product.

[4]The perturbative expansion is much better behaved than in usual thermal field theories. This is because the theory cures its infrared divergences by switching off the interactions at soft momenta. For instance, there is no need to resum daisy or Linde diagrams [31] and, to all order in perturbation theory, the free energy is given by an expansion in g^2 [26]. Similarly, the correction to the dispersion relation are $\mathcal{O}(g^4)$.

JHEP09(2000)027

noncommutative direction.[5] The fastest moving modes are the ones with longest wavelength. These are also the modes which are long lived in the thermal bath. For these it is possible to obtain the exact asymptotic behaviour by noting that the dispersion relation around $k = 0$ is

$$\omega(k) = c_0 k - \gamma k^3 + O(k^5)\,, \tag{2.14}$$

with $c_0 = \sqrt{1 + g^2\pi^2T^4\theta^2/6}$ and $\gamma = g^2\pi^4\theta^4T^6/120c_0$. This is the dispersion relation of the linearised Korteweg-deVries equation whose solution is expressed in terms of the Airy function $Ai(z)$. If we further use $F(\kappa) \approx F(0)$ for small k we can express the solution for the head of the wavetrain by

$$\Phi = \frac{F(0)}{2(3\gamma t)^{1/3}} Ai\left(\frac{x - c_0 t}{(3\gamma t)^{1/3}}\right). \tag{2.15}$$

The Airy function has oscillatory behaviour for negative argument and decays exponentially for positive argument. Thus the wavetrain decays exponentially ahead of $x = c_0 t$. Behind the wave becomes oscillatory and the Airy function matches onto the expression (2.13). In between the oscillatory region and the exponential decay there is a transition region of width proportional to $(\gamma t)^{1/3}$ around $x = c_0 t$. In this region the wavetrain has its first crest which therefore is moving with a velocity approximately given by c_0.

A condition as to when the first crest of the Airy function is a good approximation of the asymptotic behaviour of the wavetrain can be obtained as follows. The spread of the first crest gives us a length scale which should correspond to a wavelength lying in the range where (2.14) is valid. This is the case if the term of order $O(k^5)$ is small against the first two. This typically happens for $k < 1/\theta T$. That the first crest can be built up by the corresponding modes implies then $t > (\theta T)^3/\gamma$.[6] The movement of the first crest is shown in figure 2. Note that the behaviour at large x is independent of the precise form of the initial conditions.

Group velocities faster than the speed of light do also appear in conventional physics, e.g. it is well-known that this happens for light propagation in media close to an absorption line. Since the dispersive effects are however large, the group

[5] For large times t we can apply a stationary wave argument and find that Φ at the point x receives a contribution from the modes $k = \kappa$ satisfying $\omega'(\kappa) = x/t$. The asymptotic behaviour of the wave is given by [32]

$$\Phi = 2\,\mathrm{Re}\left(F(\kappa)\sqrt{\frac{2\pi}{t|\omega''(\kappa)|}}\exp\left(i\kappa x - i\omega(\kappa)t - i\frac{\pi}{4}\,\mathrm{sgn}(\omega''(\kappa))\right)\right). \tag{2.13}$$

If $\omega''(\kappa) = 0$ we have to go further in the Taylor expansion around the point of stationary phase.

[6] For very long times instabilities of the scalar excitations arising at two loop will effect the behaviour of the wavefront. Although we do not know these effects in more detail we expect that there is a regime where our one loop expression is a good approximation for long enough times.

JHEP09(2000)027

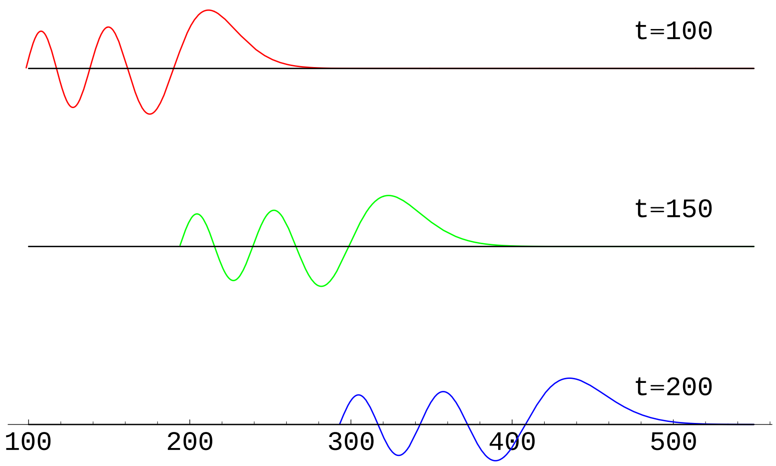

Figure 2: The movement of the first crest of a wavetrain. We have chosen $g = 0.1$, $T = 4$ and $\theta = 1$. With this choice $c_o \approx 2.2$. The crest moves with approximate speed c_o.

velocity loses its meaning as the velocity of signal transportation. In our case, it is interesting to notice that as the temperature increases, not only c_0 but also γ grows. This implies that at high temperatures the soft transverse momenta become very dispersive. In such situations it is useful to introduce the concept of a front velocity which is the velocity of the head of the wavetrain [33]. For the propagation of light in a medium it can be shown that this front velocity never exceeds the speed of light even if the group velocity can be faster than the speed of light [34]. In our case the front velocity can be defined as the velocity of the first crest of the wavetrain. According to (2.6) and (2.14) this is always bigger that the speed of light. The advance of the first crest with respect to an imagined light front is $(c_0 - 1)t$. Since its spread grows as $(\gamma t)^{1/3}$, the first crest is well defined outside the lightcone for large enough time, $t > \sqrt{\gamma/(c_0 - 1)^3}$.

The wavefront defined in the previous way is independent of the detailed form of the signal. It deserves more detailed study to answer the question of how fast can we transmit a signal. However, at least the fraction of the energy corresponding to the first crest of the wavetrain moves faster than the speed of light. Could this be a hint of violation of causality? We think this is not the case in these theories. Since the Lorentz symmetry SO$(1, 3)$ is broken to SO$(1, 1) \times$SO(2) there is no group theoretical reason for maximum velocity in the transverse directions. There exists a unique class of reference frames where the non-commutativity takes the form $[x^1, x^2] = i\theta$. If we naively perform a Lorentz boost in the say x^2 direction we find that in the new coordinate system we not only have space-space but also space-time non-commutativity. An observer being at rest in such a reference frame could measure $[x^0, x^2] = i\theta_1$ and $[x^1, x^2] = i\theta_2$. Since the non-commutativity involves three coordinates now the observer could dualize the tensor to a vector and compute its norm using the Minkowski metric. He would find $\theta_1^2 - \theta_2^2 < 0$. From this he could conclude that there is a frame in which time commutes with the other coordinates. Using this time coordinate he would also conclude that energy propagation is always forward in time. Thus the notion of an absolute, commutative time coordinate should prevent acausality to happen. Of course this still depends on the assumption that the laws of physics as formulated in the boosted reference frame are such that energy propagation is always forward in commuting time. This can not be proven at the moment since we do not know how to formulate a dynamical principle with non-commuting time coordinates. Thus, although we can not prove that the theory is causal, we can at least show that acausality is not automatically implied by superluminous effects of (2.6).

3. Wess-Zumino model

In the past section we have obtained very striking effects associated with non-commutativity in supersymmetric gauge theories. In these theories, the star-product enters in the interaction through the substitution of commutators by Moyal brackets (2.3). We will consider now the non-commutative version of the massless Wess-Zumino model, which is defined by the lagrangian [27]

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi + i\bar{\psi}\bar{\sigma}^\mu \partial_\mu \psi - g\psi * \psi\, \phi - g\bar{\psi} * \bar{\psi}\, \phi^\dagger - g^2 \phi^\dagger * \phi^\dagger\, \phi * \phi\,. \tag{3.1}$$

The interaction vertices of U(1) non-commutative theory are proportional to $\sin\frac{\tilde{p}k}{2}$, where p, k are momenta entering into the vertex, implying that the interaction switches of at low momentum. Contrary, the vertices for the Wess-Zumino model are proportional to $\cos\frac{\tilde{p}k}{2}$. The aim of this section is to study how this difference influences the dispersion relations.

At finite temperature the one loop contribution to the scalars self-energy is given by

$$\Sigma_T = 4\, g^2 \int \frac{d^3k}{(2\pi)^3} \frac{\cos^2\frac{\tilde{p}k}{2}}{k} (n_B(k) + n_F(k))\,. \tag{3.2}$$

We have ignored terms proportional to g^2P^2 for the same reasons already explained in the previous section. We obtain the following dispersion relation

$$\omega^2 = p^2 + \frac{g^2T^2}{4} + \frac{g^2T}{2\pi\tilde{p}} \tanh\frac{\pi T\tilde{p}}{2}\,. \tag{3.3}$$

Contrary to (2.6), the planar and non-planar corrections, second and third term in (3.3) respectively, add up instead of subtracting. As a result all the modes down to zero momentum acquire a mass due to thermal effects. At small temperature the minimum is at zero momentum. Since the non-planar contribution tends to the planar one in the small $\tilde{p}$ limit, the thermal mass is $m^2 \approx 2\, m^2_{\text{planar}} = g^2T^2/2$. At high temperature the system behaves quite differently. There is a temperature

$$T_0 = \left(\frac{48}{\pi^2 g^2 \theta^2}\right)^{1/4}\,, \tag{3.4}$$

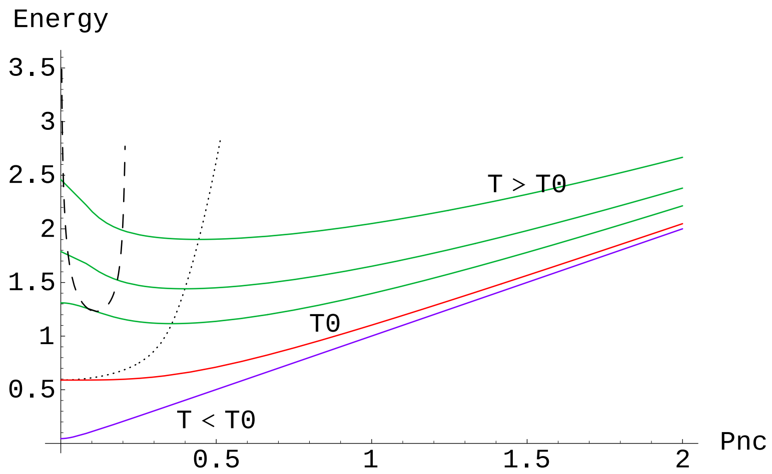

Figure 3: Plot of ω as a function of $p_\perp$ for different temperatures and $p_3 = 0$ ($g^2 = 0.1$, $\theta = 1$). For $T > T_0$ the dispersion relation develops a dip. The dotted line locates the position of the minimum.

such that for $T > T_0$ the lowest energy mode lies at $p_\perp^{\rm min} > 0$, as can be seen in figure 3. For large temperatures we find

$$p_\perp^{\rm min} \approx \left(\frac{g^2 T}{4\pi\theta}\right)^{1/3} . \tag{3.5}$$

Temperatures bigger than T_0 implies $T^2\theta \gtrsim 1/g$. At weak coupling, this effect is seen at temperatures much higher than the expected scale for the onset of non-commutative effects $T^2\theta > 1$.

As T increases and for $p_\perp > p_\perp^{\rm min}$ the thermal mass tends to $m^2_{\rm planar} = g^2T^2/4$. Clearly, because $p_\perp^{\rm min} \propto T^{1/3}$, at sufficiently high temperature the group velocity $\partial\omega/\partial p_\perp < -1$. This is indicated in figure 3 by the region inside the dashed line. As discussed in the previous section in footnote 5 we can apply a stationary phase approximation. Note that this is valid whenever the imaginary part of the dispersion relation vanishes.[7] This is the case to the approximation we are working. Thus the modes with $\partial\omega/\partial p_\perp < -1$ contribute to a wave at $x = t\omega'$ outside the lightcone.

The dip in the dispersion relation has also rather interesting consequences for the phase structure of the theory. Consider for instance a gas which is not only hot but also carries some net global charge. The massless Wess-Zumino model has a U(1) R-symmetry under which the the scalar and Weyl fermion have charge $+2/3$ and $-1/3$ respectively. A non-zero charge density Q/V is imposed by introducing chemical potentials for the bosons and fermions. In chemical equilibrium, the

[7] $\partial\omega/\partial p < -1$ also happens for propagation of light in a dispersive medium close to an absorption line. However the imaginary part of the dispersion relation does not vanish and this is the reason why one has to apply a more accurate saddle point method [34]. In this case the saddle point evaluation shows that the speed of light is never exceeded.

fermions and bosons chemical potential are related, $\mu_B = 2\mu_F$, so we concentrate on the bosons. For fixed external charge the chemical potential and the temperature are not independent, $\mu_B = \mu_B(T)$. Usually, if the bosons were massive in the non-interacting limit, they would undergo Bose-Einstein condensation at some critical temperature $T_{BC} \propto (Q/V)^{2/3}/m_B$ where m_B is the boson mass. Operationally, this is the temperature at which the boson chemical potential $|\mu_B| = m_B$. Obviously, if $m_B \to 0$, the charge is always in the zero momentum condensate i.e. $T_{BC} \to \infty$. Interactions change this as the bosons get a mass $m_B(T) \sim gT$ in a thermal bath, but the reasoning is essentially the same with $|\mu_B| = m_B(T)$ [35]. Now consider our non-commutative theory. Below T_0 the theory has a behaviour very much analogous to a conventional field theory at finite temperature. Above T_0 on the other hand, the minimum of the dispersion relation is at non-zero transverse momentum. Bosons with such a dispersion relation cannot undergo Bose-Einstein condensation. This is because the chemical potential is bounded by the minimum of the boson dispersion relation, $|\mu_B(T)| \leq \omega(p_\perp^{\rm min}, T)$. For $|\mu_B(T)| \to \omega(p_\perp^{\rm min}, T)$ the integral for the global charge density Q/V

$$Q/V \sim \int d^3p \frac{1}{\exp\beta(\omega(p) - \omega(p_\perp^{\rm min})) - 1} \sim \int dp_\perp \frac{p_\perp}{|p_\perp - p_\perp^{\rm min}|} \tag{3.6}$$

diverges when $p_\perp^{\rm min} \neq 0$.[8] We thus expect that, at high densities T_0 is the maximal critical temperature for Bose-Einstein condensation.

4. Gauge theories at finite density

We have studied the behaviour of non-commutative theories in a thermal bath. Temperature allowed us to probe the theory at different scales and observe the deviations with respect to the ordinary behaviour. There are however other ways of probing the system. It is interesting to analyse how much the effects we have obtained depend on the fact of having a thermal bath or are reproduced in other physical settings. In particular, we can consider theories at finite density. The effects of non-vanishing chemical potential have been studied in [18] in a non-relativistic, non-commutative field theory at finite temperature. It turned out that the effects of non-commutativity depend strongly on the value of the chemical potential. In the previous section we have analysed the Wess-Zumino model at finite temperature and density. There again the behaviour of the system with respect to Bose-Einstein condensation depended strongly on the density. We want to study now a situation where the only scale present is provided by the chemical potential, namely finite density at zero temperature.

[8]That the dispersion relation prevents the condensation of zero modes is analogous to the effect discussed in [25]. However here the chemical potential is not an independent parameter that can be varied to go from the disordered to the ordered phase, but a constraint which adjusts itself so as to insure charge conservation.

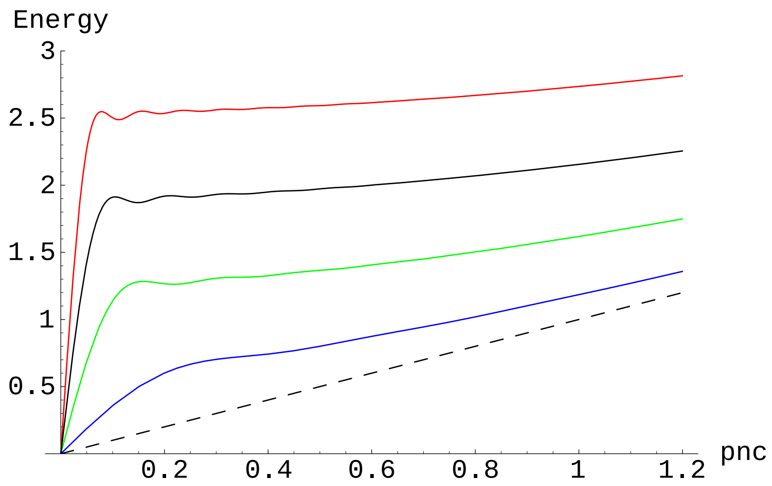

Figure 4: ω as a function of $p_\perp$ for different values of the chemical potential ($p_3 = 0, g = 0.1, \theta = 1$).

Let us take as an example $\mathcal{N} = 2$ non-commutative gauge theory. The spectrum consists of a vector field, two Weyl spinors and a complex scalar. Ordinary $\mathcal{N} = 2$ Yang-Mills theory has a non-anomalous $\mathrm{SU}(2)_R$ symmetry under which vector and scalars are singlets and the two $\mathcal{N} = 2$ gluinos transform as a doublet. This is still valid for the non-commutative version. There is a $\mathrm{U}(1) \in \mathrm{SU}(2)_R$ under which the gluinos λ and ψ transform with charges 1 and -1 respectively. A non-zero density of the associated conserved charge is imposed by introducing a chemical potential μ. Positive chemical potential corresponds to an excess of $(\lambda, \bar{\psi})$ quanta over $(\psi, \bar{\lambda})$ quanta in the system, and vice-versa for negative chemical potential. To be definite we consider μ positive.

For simplicity we will work with a $\mathrm{U}(1)$ $\mathcal{N} = 2$ non-commutative theory. The one loop contribution the the scalar self-energy due to a finite fermion density is given by

$$\Sigma = 8g^2 \int \frac{d^3k}{(2\pi)^3} \frac{\sin^2 \frac{\tilde{p}k}{2}}{k} \Theta(\mu - k)\,, \tag{4.1}$$

where $\Theta(x)$ is the step-function. We obtain the following modified dispersion relation

$$\omega^2 = p^2 + \frac{g^2\mu^2}{\pi^2} - \frac{4g^2}{\pi^2\tilde{p}^2} \sin^2 \frac{\mu\tilde{p}}{2}\,. \tag{4.2}$$

The dispersion relation (4.2) is similar to the one obtained for $\mathcal{N} = 4$ gauge theories in a thermal bath, with the chemical potential μ playing the role of temperature. The spectrum is gapless as can be seen in figure 4. For small transverse momentum we have group velocities bigger than one,

$$\omega^2 = \left(1 + \frac{g^2\theta^2\mu^4}{12\pi^2}\right) p_\perp^2\,. \tag{4.3}$$

For chemical potential $\mu^2 \gg 1/g\theta$ and $p_\perp > p_\perp^0$,

$$p_\perp^0 \approx \left(\frac{2g^2\mu}{\pi^3\theta}\right)^{1/3}, \tag{4.4}$$

the scalar modes have a mass $m \approx g\mu/\pi$. A new feature of (4.2) with respect (2.6) is the appearance of a very dispersive region at $2\pi/\mu\theta < p_\perp < p_\perp^0$. In this region the $\sin^2$ term on (4.2) dominates and ω has a series of maxima and minima with period $2\pi/\mu\theta$. The amplitude of the oscillations grows with the density, but they are always of small relative size with respect to the first maximum.

5. Discussion

We have investigated the dispersion relations for scalar excitations in non-commutative supersymmetric theories at finite temperature and density. We found several striking new phenomena. In particular, wavefronts can travel faster than the speed of light. In the Wess-Zumino model, above some temperature, there is a dip in the dispersion relation and the minimum of energy is at non zero transverse momentum.

These features are in essence already present in noncommutative theories in vacuum and we expect them to be quite generic consequences of spatial non-commutativity. At one loop, non-planar diagrams give rise to pole-like divergences in the dispersion relations and to the now famous issue of UV/IR mixing. While in supersymmetric theories these puzzling divergences are absent in vacuum, they reappear in disguise in presence of a thermal bath. Finite temperature or density not only breaks supersymmetry but also provides a natural cut-off which allows to investigate the issue of UV/IR mixing in a controlled setting. In particular, the new effects we observed are more pronounced as we increase temperature or a chemical potential. Since these parameters act as UV-regulators our findings are a direct consequence of the UV/IR mixing in non-commutative field theories.

We have found dispersive wavefronts travelling faster than the speed of light. As we tried to argue this is a priori not inconsistent since these theories are not Lorentz invariant. The $\mathcal{N} = 4$ theory can be realized as the decoupling limit of a D3-brane in a B-field background in string theory. Since string theory is Lorentz invariant it is puzzling how velocities faster than the speed of light can appear. One possibility is that the tower of additional string states modifies the dispersion relation in a convenient way. This question is currently under investigation.

We have limited ourself to one loop, real tadpole corrections. At two loop, we expect scalar excitations in a thermal bath to be also unstable, both because of Landau damping and particle pair creation. While these effects are subleading in g, the instabilities will also affect the long time behaviour of the propagating wavefront. It would be interesting to investigate this point further.

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